J Glob Optim (2008) 42:111–120
DOI 10.1007/s10898-007-9253-y

Iterative solving of variational inclusions under Wijsman perturbations

Michel H. Geoffroy · Célia Jean-Alexis · Alain Piétrus

Received: 5 March 2007 / Accepted: 6 October 2007 / Published online: 24 October 2007

Abstract We prove that the metric regularity of set-valued mappings is stable under some Wijsman-type perturbations. Then, we solve a variational inclusion viewed as a limit-problem using assumptions on a sequence of associated problems. Finally, we apply our results to classical methods for solving variational inclusions.

Keywords Set-valued mappings · Aubin continuity · Metric regularity · Wijsman convergence · Point-based-approximation

AMS Subject Classifications (2000) 49J53 · 65K10 · 54A20

1 Introduction

In [9], Geoffroy and Piétrus presented a general iterative procedure for solving variational inclusions of the form

$$0 \in f(x) + F(x), \tag{1}$$

where $f: X \to Y$ is a function acting between two Banach spaces and admitting a so-called (n, α)-*point-based approximation* $A: X \times X \to Y$, while F stands for a set-valued mapping. To solve this inclusion the authors considered a method which generates a sequence x_k of iterates by solving the following approximate subproblem:

$$0 \in A(x_k, x_{k+1}) + F(x_{k+1}), \tag{2}$$

M. H. Geoffroy (✉) · C. Jean-Alexis · A. Piétrus
Laboratoire AOC, Département de Mathématiques et Informatique, Campus de Fouillole,
Université des Antilles et de la Guyane, 97159 Pointe-a-Pitre, France
e-mail: michel.geoffroy@univ-ag.fr

C. Jean-Alexis
e-mail: celia.jean-alexis@univ-ag.fr

A. Piétrus
e-mail: alain.pietrus@univ-ag.fr

where the mapping A replace the function f. Let us point out that, thanks to the properties of the approximation A, it has been proved that several existing methods for solving variational inclusions are subsumed within the iteration (2). For instance, when f is smooth, it recovers both a Newton-type method (due to Dontchev [3]) for solving generalized equations:

$$0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + F(x_{k+1}),$$

and the following cubically convergent method (see in [8,10])

$$0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + \frac{1}{2}\nabla^2 f(x_k)(x_{k+1} - x_k)^2 + F(x_{k+1}).$$

Before going further let us recall the definition of a (n, α)-point-based approximation.

Definition 1.1 Let f be a function from an open subset Ω of a metric space (X, d) to a normed linear space Y and fix $n \in \mathbb{N}^*$ and $\alpha > 0$. Consider a function $A: \Omega \times \Omega \to Y$ and a scalar k such that for each u and v in Ω both of the following assertions hold:

(a) $\| f(v) - A(u, v)\| \leq \dfrac{k}{\pi_{n,\alpha}} d(u, v)^{n+\alpha}$, where $\pi_{n,\alpha} = \prod_{i=1}^{n} (\alpha + i)$;
(b) The function $A(u, \cdot) - A(v, \cdot)$ is Lipschitzian on Ω with constant $\kappa(k)d(u, v)^{\alpha}$, where κ is a positive function of k.

Then, A is said to be a (n, α)-point-based approximation ((n, α)-PBA for short) on Ω for f with modulus k.

Definition 1.1 generalizes the concept of point-based approximation presented by Robinson (see [15]), actually, it agrees with Robinson's definition of PBA when n and α are equal to 1. Note that one can find both smooth and nonsmooth functions satisfying assertions (a) and (b) in Definition 1.1, i.e., admitting a (n, α)-PBA (see [9] for details).

Moreover, it is proved in [9] that the method (2) is super-linearly convergent whenever the function f and the set-valued mapping F satisfy some mild conditions. Indeed, making the following assumptions,

(H1) *F has closed graph;*
(H2) *f admits a (n, α)-point-based approximation with modulus k, denoted by A, on some open neighborhood Ω of $\bar{x}$;*
(H3) *The set-valued map $[\, A(\bar{x}, \cdot) + F(\cdot)\,]^{-1}$ is Aubin continuous at $(0, \bar{x})$, or equivalently $A(\bar{x}, \cdot) + F(\cdot)$ is metrically regular at $\bar{x}$ for 0*

we have,

Theorem 1.2 [9] *Let $\bar{x}$ be a solution of $0 \in f(x) + F(x)$ and suppose that the assumptions (H1)–(H3) are satisfied. Then for every $C > \frac{Mk}{\pi_{n,\alpha}}$ one can find $\delta > 0$ such that for every starting point $x_0 \in B_\delta(\bar{x})$ there exists a sequence x_k, defined by (2), which satisfies*

$$\|x_{k+1} - \bar{x}\| \leq C\|x_k - \bar{x}\|^{n+\alpha}. \tag{3}$$

Inclusion (1), and variational inclusions in general, serve as a general tool for describing and solving various problems in a unified manner. As an illustration, we consider the so-called *feasibility problems*: let X and Y be Banach spaces and let $K \subset Y$ be a closed convex cone. Considering a function $f: X \to Y$, a closed convex set $C \subset X$ and a point $y \in Y$, the feasibility problem consists in finding $x \in C$ such that

$$y \in f(x) + K. \tag{4}$$

We show how such a problem fits the setting of Theorem 1.2. We set $y := 0$ then relation (4) can be rewritten in the form $0 \in F(x)$ where

$$F(x) = \begin{cases} f(x) + K & \text{if } x \in C, \\ \emptyset & \text{otherwise.} \end{cases}$$

In addition, according to [4], when f is strictly differentiable the mapping F is metrically regular at $\bar{x}$ for 0 (where $(\bar{x}, 0)$ is in the graph of F) if and only if the mapping L, given by

$$L(x) = \begin{cases} f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) + K & \text{if } x \in C, \\ \emptyset & \text{otherwise,} \end{cases}$$

is metrically regular at $\bar{x}$ for 0. Note that whenever the Fréchet derivative of f is Lipschitz continuous, the mapping $A\colon (u, v) \mapsto f(u) + \nabla f(u)(v - u)$ is a (1, 1)-PBA for f. Since L has closed and convex graph, from the works of Ursescu [18] and Robinson [14] we know that the mapping $L(\cdot) = A(\bar{x}, \cdot) + K$ is metrically regular at $\bar{x}$ for 0 if and only if 0 lies in the interior of the range of L. When $K = \{0\}^r \times \mathbb{R}^m_+$ where $\{0\}^r$ is the origin in $\mathbb{R}^r$ and $C = \mathbb{R}^n$ then the mapping F represents a system of equalities and inequalities in finite dimensions. In that particular case the Robinson–Ursescu condition is equivalent to the well-known *Mangasarian-Fromovitz property* (see, e.g., [7]).

Our purpose here is to solve the variational inclusion (1) when F is the limit of a sequence of set-valued mappings F_n. More precisely, we consider a sequence of *metrically regular* set-valued mappings $F_n\colon X \rightrightarrows Y$ converging in a certain sense to F and we study the stability of the regularity properties of the sequence F_n. To describe the behavior of the sequence F_n we use the notion of *Wijsman convergence* which appears to be very well suited to treat this kind of problems involving metric regularity.

We show that the metric regularity of the elements of the sequence F_n is inherited by the function F and this allows us to apply Theorem 1.2 to solve (1).

The content of this paper is as follows. In Sect. 2 we present some background material on metric regularity and variational convergences. Next, in Sect. 3, we prove that the metric regularity of set-valued mappings is stable under some Wijsman-type perturbations. Then we are able to solve a limit-problem (involving Wijsman convergence) using assumptions on a sequence of associated problems. Finally, in Sect. 4, we apply our results to classical methods for solving variational inclusions.

2 Background material

Throughout, X and Y are Banach spaces, we indicate by $F : X \rightrightarrows Y$ a set-valued mapping from X into the subsets of Y. The set $\operatorname{gph} F = \{(x, y) \in X \times Y \mid y \in F(x)\}$ is the graph of F and F^{-1} stands for the inverse of F defined as $x \in F^{-1}(y) \Leftrightarrow y \in F(x)$. We denote by $d(x, C)$ the distance from a point x to a set C, that is, $d(x, C) = \inf_{y \in C} \|x - y\|$. The closed unit ball is denoted by $I\!B$ while $I\!B_r(a)$ stands for the closed ball of radius r centered at a.

Definition 2.1 (*Metric regularity*) A set-valued map $F\colon X \rightrightarrows Y$ is metrically regular at x_0 for y_0 if $y_0 \in F(x_0)$ and there exist positive constants a, b and κ such that:

$$d(x, F^{-1}(y)) \le \kappa d(y, F(x)), \quad \forall\, x \in I\!B_a(x_0),\ y \in I\!B_b(y_0). \tag{5}$$

The infimum of κ for which (5) holds is the *regularity modulus* denoted $\operatorname{reg} F(x_0 \mid y_0)$; the case when F is not metrically regular at x_0 for y_0 corresponds to $\operatorname{reg} F(x_0 \mid y_0) = \infty$.

The inequality (5) has direct use in providing an estimate for how far a point x is from being a solution to the variational inclusion $F(x) \ni y$; the expression $d(y, F(x))$ measures the residual when $F(x) \not\ni y$. Smaller values of κ correspond to more favorable behavior. The metric regularity of a mapping F at $\bar{x}$ for $\bar{y}$ is known to be equivalent to the Aubin property of the inverse F^{-1} at $\bar{y}$ for $\bar{x}$ (see, e.g., [16]). Recall that a set-valued map $\Gamma : Y \rightrightarrows X$ has the Aubin property at $(\bar{y}, \bar{x})$ (see [1]) if there exist positive constants κ, a and b such that

$$e(\Gamma(y') \cap I\!B_a(\bar{x}), \Gamma(y)) \leq \kappa \|y' - y\| \quad \text{for all } y, y' \in I\!B_b(\bar{y}), \tag{6}$$

where $e(A, B)$ denotes the excess from a set A to a set B and is defined as $e(A, B) = \sup_{x \in A} d(x, B)$. For more details on metric regularity and applications to variational problems one can refer to [6,11,12] and the monograph [16].

Definition 2.2 (*Equi-metric regularity*) We say that a sequence $F_n : X \rightrightarrows Y$ of set-valued mappings is equi-metrically regular at x_n for y_n if $y_n \in F_n(x_n)$ for all n and there exist positive constants (which do not depend on n) a, b and κ such that

$$d(x, F_n^{-1}(y)) \leq \kappa d(y, F_n(x)), \quad \forall\, x \in I\!B_a(x_n),\ y \in I\!B_b(y_n).$$

A central result in the theory of metric regularity is the Lyusternik–Graves theorem which roughly says that the metric regularity is stable under perturbations of order higher than one. In the general form of this theorem we present next and which is from [5], we use the following convention: we say that a set $C \subset X$ is locally closed at $z \in C$ if there exists $a > 0$ such that the set $C \cap I\!B_a(z)$ is closed. Recall also that the Lipschitz modulus of a function f at a point $\bar{x} \in \operatorname{int} \operatorname{dom} f$ is defined as

$$\operatorname{lip}(f; \bar{x}) = \limsup_{\substack{x, x' \to \bar{x} \\ x \neq x'}} \frac{\|f(x) - f(x')\|}{\|x - x'\|}.$$

Theorem 2.3 ([5], Extended Lyusternik–Graves) *Consider a mapping $F\colon X \rightrightarrows Y$ and any $(x_0, y_0) \in \operatorname{gph} F$ at which* gph F *is locally closed. Consider also a function $g\colon X \to Y$ with $x_0 \in \operatorname{int} \operatorname{dom} g$. If* $\operatorname{reg} F(x_0 \mid y_0) < \kappa < \infty$ *and* $\operatorname{lip}(g; x_0) < \delta < \kappa^{-1}$*, then*

$$\operatorname{reg}(g + F)(x_0 \mid g(x_0) + y_0) \leq (\kappa^{-1} - \delta)^{-1}.$$

Now, we briefly present the convergence notions we shall use in the sequel. The lower and upper limits of a sequence A_n of subsets of a normed space, with unit ball $I\!B$, are defined by

$$\liminf_n A_n := \bigcap_{\varepsilon > 0} \bigcup_{N > 0} \bigcap_{n \geq N} (A_n + \varepsilon I\!B) \ \text{ and } \ \limsup_n A_n := \bigcap_{\varepsilon > 0} \bigcap_{N > 0} \bigcup_{n \geq N} (A_n + \varepsilon I\!B).$$

A useful alternative formulation (in normed spaces) is given by

$$\begin{aligned} \liminf_n A_n &= \{x \in X \mid \limsup_{n \to \infty} d(x, A_n) = 0\} \\ &= \{x \in X \mid \exists x_n \in A_n \text{ with } x_n \to x\}; \end{aligned}$$

$$\begin{aligned} \limsup_n A_n &= \{x \in X \mid \liminf_{n \to \infty} d(x, A_n) = 0\} \\ &= \{x \in X \mid \exists n_1 < n_2 < \cdots \text{ in } \mathbb{N},\ \exists x_{n_k} \in A_{n_k} \text{ with } x_{n_k} \to x\}. \end{aligned}$$

Definition 2.4 A sequence of subsets A_n is said to *set-converge* to a subset A, written $A_n \to A$, provided $\limsup_n A_n \subset A \subset \liminf_n A_n$.

Set convergence in this sense is known more specifically as Painlevé–Kuratowski convergence. A sequence A_n of subsets of X is said to be *lower set-convergent* to A if $A \subset \liminf_n A_n$ and *upper set-convergent* to A if $\limsup_n A_n \subset A$. Obviously, a sequence A_n set-converges to A if and only if it is both lower and upper set-convergent to A.

Definition 2.5 A sequence of subsets A_n is said to *Wijsman-converge* to a subset A if for every $u \in X$, $\lim_{n\to\infty} d(u, A_n) = d(u, A)$.

We will say that a sequence of subsets A_n is *lower Wijsman-convergent* to A provided $d(u, A) \le \liminf_n \, d(u, A_n)$, for all $u \in X$. Similarly, a sequence of subsets will be said to be *upper Wijsman-convergent* to A if $\limsup_n \, d(u, A_n) \le d(u, A)$, for all $u \in X$.

It is well known that the Wijsman-convergence of a sequence A_n to A implies the set-convergence of this sequence to A. Nevertheless, we give here a more precise statement involving both lower and upper Wijsman convergences.

Lemma 2.6 *Let A_n be a sequence of subsets of X. Then,*

(1) *If A_n lower Wijsman-converges to a closed subset A of X then A_n upper set-converges to A.*
(2) *If A_n upper Wijsman-converges to a subset A of X then A_n lower set-converges to A.*

Proof (1) Let A_n be a sequence of subsets lower Wijsman-converging to A. Then

$$d(u, A) \le \liminf_n \, d(u, A_n), \quad \text{for all } u \in X. \tag{7}$$

Take $x \in \limsup_n A_n$, then $\liminf_n d(x, A_n) = 0$, and by (7), $d(x, A) = 0$, i.e., $x \in A$. Thus, A_n upper set-converges to A.

(2) Let A_n be a sequence of subsets upper Wijsman-converging to A. Take $x \in A$ then we have $\limsup_n \, d(x, A_n) = 0$, i.e., $x \in \liminf_n \, A_n$. □

It is worth pointing out that Wijsman convergence is one of the weakest variational convergences. For instance, it is forced by the convergences of Hausdorff and Attouch-Wets and as well by the linear convergence. For a comprehensive account on variational convergences, the reader could refer to the monograph by Beer [2], and to the survey paper of Sonntag and Zălinescu [17].

3 Wijsman perturbations

Consider a set-valued mapping $T\colon X \rightrightarrows Y$ such that $T^{-1}(0)$ is a nonempty closed subset of X and let $T_n\colon X \rightrightarrows Y$ be a sequence of set-valued mappings.

Lemma 3.1 *Let x_n be a sequence in X such that $T_n(x_n) \ni 0$, eventually. If x_n converges to $x \in X$ and $T_n^{-1}(0)$ upper set-converges to $T^{-1}(0)$ then $T(x) \ni 0$.*

Proof From the upper set-convergence of $T_n^{-1}(0)$ to $T^{-1}(0)$ we have

$$\limsup_n \, T_n^{-1}(0) \subset T^{-1}(0).$$

Hence, to prove that $T(x) \ni 0$ it suffices to show that $x \in \limsup_n T_n^{-1}(0)$. This is immediate since $x_n \to x$ and $\liminf_n d(x, T_n^{-1}(0)) \le \liminf_n \|x - x_n\|$. □

Remark 3.2 Note that the lower set-convergence of $T_n^{-1}(0)$ to $T^{-1}(0)$ ensures that for any $\bar{x} \in T^{-1}(0)$ there exists a sequence x_n converging to $\bar{x}$ and such that $T_n(x_n) \ni 0$.

The following result expresses that the metric regularity of a set-valued mapping is stable under some weak Wijsman perturbations of the ranges of this mapping and its inverse.

Proposition 3.3 *Let $\bar{x} \in X$ and assume that both assertions* (i) *and* (ii) *below hold.*

(i) *$T_n(x)$ upper Wijsman-converges to $T(x)$ for all x close to $\bar{x}$;*
(ii) *$T_n^{-1}(y)$ lower Wijsman-converges to $T^{-1}(y)$ for all y close to* 0.

Let $\bar{x}_n$ be a sequence converging to $\bar{x}$ and such that $T_n(\bar{x}_n) \ni 0$ eventually. If, in addition, the sequence T_n is equi-metrically regular at $\bar{x}_n$ for 0 *then T is metrically regular at $\bar{x}$ for* 0 *and in particular* $0 \in T(\bar{x})$.

Proof Thanks to assertion (ii), combining Lemmas 2.6 and 3.1, we obtain that $T(\bar{x}) \ni 0$. From the equi-metric regularity of the sequence T_n at $\bar{x}_n$ for 0 there exist positive constants a, b and κ such that

$$d(x, T_n^{-1}(y)) \le \kappa\, d(y, T_n(x)) \quad \text{for all } x \in I\!B_a(\bar{x}_n),\ y \in I\!B_b(0). \tag{8}$$

Take $\alpha < a$ and pick an arbitrary point z in $I\!B_\alpha(\bar{x})$. Then, $\|z - \bar{x}_n\| \le \|z - \bar{x}\| + \|\bar{x} - \bar{x}_n\|$. Since $\bar{x}_n \to \bar{x}$, for n large enough, we have $\|z - \bar{x}_n\| \le a$. Hence, $I\!B_\alpha(\bar{x}) \subset I\!B_a(\bar{x}_n)$, eventually. Thus, thanks to relation (8) we get, for n large enough,

$$d(x, T_n^{-1}(y)) \le \kappa\, d(y, T_n(x)) \quad \text{for all } x \in I\!B_\alpha(\bar{x}),\ y \in I\!B_b(0).$$

Then,

$$\limsup_n d(x, T_n^{-1}(y)) \le \kappa \ \limsup_n d(y, T_n(x)) \quad \text{for all } x \in I\!B_\alpha(\bar{x}),\ y \in I\!B_b(0).$$

Making α and b smaller if necessary and using assertions (i) and (ii) we obtain

$$d(x, T^{-1}(y)) \le \kappa\, d(y, T(x)) \quad \text{for all } x \in I\!B_\alpha(\bar{x}),\ y \in I\!B_b(0),$$

that is, T is metrically regular at $\bar{x}$ for 0. □

The following statement, that we will need to prove our main result, asserts that the Aubin continuity of the mapping $(f + F)^{-1}$ around some point of its graph is equivalent to the Aubin continuity (at the same point) of the inverse of some approximation of $f + F$ involving a (n, α)-PBA.

Proposition 3.4 *Let $f : X \to Y$ be a function admitting a (n, α)-point-based approximation A with modulus k on some open neighborhood of a point $\bar{x} \in X$. Consider in addition a set-valued mapping $F : X \rightrightarrows Y$, then the following are equivalent*

(i) *the mapping $(f + F)^{-1}$ is Aubin continuous around $(\bar{y}, \bar{x})$,*
(ii) *the mapping $(A(\bar{x}, \cdot) + F(\cdot))^{-1}$ is Aubin continuous around $(\bar{y}, \bar{x})$.*

Proof According to [4, Corollary 2] proving the equivalence in Proposition 3.4 reduces to showing that the mapping $g: X \to Y$, defined by $g(x) = f(x) - A(\bar{x}, x)$, is strictly stationary at $\bar{x}$, i.e., for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|g(x) - g(x')\| \leq \varepsilon \|x - x'\| \quad \text{for all } x, x' \in I\!B_\delta(\bar{x}).$$

Fix $\varepsilon > 0$ and take a positive number δ such that

$$\delta \leq \min \left\{ \left(\frac{\varepsilon \pi_{n,\alpha}}{2^{n+\alpha} k} \right)^{\frac{1}{n-1+\alpha}}, \left(\frac{\varepsilon}{2\kappa(k)} \right)^{\frac{1}{\alpha}} \right\},$$

where $\kappa(k)$ is the function from the Lispchitz constant of the mapping $A(u, \cdot) - A(v, \cdot)$ (see Definition 1.1). Taking arbitrary points $x, x' \in I\!B_\delta(\bar{x})$, we have,

$$\begin{aligned} \|g(x) - g(x')\| &= \|f(x) - A(\bar{x}, x) - f(x') + A(\bar{x}, x')\| \\ &= \|f(x) - A(x', x) + A(x', x) - A(\bar{x}, x) - A(x', x') + A(\bar{x}, x')\|, \end{aligned}$$

since assertion (a) in Definition 1.1 yields $f(x') = A(x', x')$. Hence we get,

$$\begin{aligned} \|g(x) - g(x')\| &\leq \frac{k}{\pi_{n,\alpha}} \|x - x'\|^{n+\alpha} + \|(A(x', x) - A(\bar{x}, x)) - (A(x', x') - A(\bar{x}, x'))\| \\ &\leq \frac{k}{\pi_{n,\alpha}} \|x - x'\|^{n+\alpha} + \kappa(k) \|x - \bar{x}\|^\alpha \|x - x'\| \\ &\leq \left(\frac{k}{\pi_{n,\alpha}} (2\delta)^{n-1+\alpha} + \kappa(k) \delta^\alpha \right) \|x - x'\| \\ &\leq \varepsilon \|x - x'\|. \end{aligned}$$

□

From now on, $f: X \to Y$ stands for a continuous function and $F: X \rightrightarrows Y$ is a set-valued mapping with closed graph. We consider a sequence of set-valued mappings $F_n: X \rightrightarrows Y$ such that the following assumption holds:

(A_1) There is a sequence $\bar{x}_n$ converging to some point $\bar{x} \in X$ and satisfying

$$f(\bar{x}_n) + F_n(\bar{x}_n) \ni 0, \text{ eventually.}$$

Moreover, we assume that F is the limit of the sequence F_n in the following sense:

(A_2) $F_n(x)$ upper Wijsman-converges to $F(x)$ for all x close to $\bar{x}$.
(A_3) $(f + F_n)^{-1}(y)$ lower Wijsman-converges to $(f + F)^{-1}(y)$ for all y close to 0.

Theorem 3.5 *Assume that the sequence F_n is equi-metrically regular at $\bar{x}_n$ for $-f(\bar{x}_n)$ with a growth constant $\kappa > \sup_n \operatorname{reg} F_n(\bar{x}_n \mid -f(\bar{x}_n))$. If, in addition, the function f admits a (n, α)-PBA on some open neighborhood of $\bar{x}$ and is such that $\operatorname{lip}(f; \bar{x}) < \kappa^{-1}$, then under assumptions (A_1)–(A_3), there exists a sequence x_k generated by (2) and converging super-linearly with order $(n + \alpha)$ to $\bar{x}$ which is a solution to $f(x) + F(x) \ni 0$.*

Proof According to Theorem 1.2, to obtain the existence of the sequence x_n, we have to prove that the mapping $[A(\bar{x}, \cdot) + F(\cdot)]^{-1}$ is Aubin continuous at $(0, \bar{x})$ which, thanks to Proposition 3.4, is equivalent to showing that the mapping $(f + F)^{-1}$ is Aubin continuous at $(0, \bar{x})$.

To this end, we apply Proposition 3.3 with $T := f + F$ and $T_n := f + F_n$. Since gph F is closed, the continuity of f yields that gph$(f + F)$ is also closed. It follows that $(f + F)^{-1}(y)$

is closed for any $y \in Y$ and in particular $(f+F)^{-1}(0)$ is a closed subset of X. Then, keeping in mind assumption (A_3) and applying Lemmas 2.6 and 3.1, we get that $\bar{x}$ is a solution to (1).

Moreover, it is clear that the upper Wijsman-convergence of $F_n(x)$ to $F(x)$ for all x close to $\bar{x}$ implies the upper Wijsman-convergence of $(f+F_n)(x)$ to $(f+F)(x)$ for the same points x.

Then, it remains to show that $f+F_n$ is equi-metrically regular at $\bar{x}_n$ for 0. The sequence F_n is equi-metrically regular at $\bar{x}_n$ for $-f(\bar{x}_n)$ with a growth constant κ such that $\kappa > \sup_n \operatorname{reg} F_n(\bar{x}_n \mid -f(\bar{x}_n))$. Thus, to apply the extended Lyusternik–Graves theorem (Theorem 2.3), we have to verify that, eventually, one has $\operatorname{lip}(f;\bar{x}_n) < \kappa^{-1}$. This is an immediate consequence of the fact that the function f is such that $\operatorname{lip}(f;\bar{x}) < \kappa^{-1}$ together with the convergence of the sequence $\bar{x}_n$ to $\bar{x}$. Hence the mapping $f+F_n$ is equi-metrically regular at $\bar{x}_n$ for 0 and applying Proposition 3.3 we obtain that $f+F$ is metrically regular at $\bar{x}$ for 0, i.e., $(f+F)^{-1}$ is Aubin continuous at $(0,\bar{x})$.

Finally, Theorem 1.2 gives us the existence of a sequence x_k generated by (2) and converging super-linearly to $\bar{x}$. □

4 Applications

In the case when f is a smooth functional we are able to provide two direct applications of Theorem 3.5. Throughout, $F\colon X \rightrightarrows Y$ denotes a set-valued mapping with closed graph. While $F_n\colon X \rightrightarrows Y$ is a sequence of set-valued mappings satisfying assumption (A_1) and approximating F in the sense of assertions (A_2) and (A_3).

4.1 A Newton-type method

When the function f is Fréchet differentiable near the point $\bar{x}$ and is such that ∇f is locally Lipschitz near this point, Dontchev [3], considers the following Newton-type method to solve the inclusion $0 \in f(x) + F(x)$:

$$0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + F(x_{k+1}). \tag{9}$$

He shows that the iteration (9) generates a sequence x_n which is quadratically convergent to a solution to $0 \in f(x) + F(x)$ whenever $(f+F)^{-1}$ is Aubin continuous at $(0,\bar{x})$. Note that a variant of this method have also been considered in [13] where the derivative ∇f is no longer Lipschitz continuous but Hölder continuous.

Corollary 4.1 *Assume that the sequence F_n is equi-metrically regular at $\bar{x}_n$ for $-f(\bar{x}_n)$ with a growth constant $\kappa > \sup_n \operatorname{reg} F_n(\bar{x}_n \mid -f(\bar{x}_n))$. If, in addition, the function f is Fréchet differentiable in an open neighborhood Ω of $\bar{x}$ and its derivative is Lipschitz continuous on Ω and bounded by a constant $M < \kappa^{-1}$ then, under assumptions* (A_1)–(A_3)*, there exist a sequence x_k generated by* (9) *and a positive constant C such that*

$$\|x_{k+1} - \bar{x}\| \le C\|x_k - \bar{x}\|^2,$$

i.e., the sequence x_k converges quadratically to $\bar{x}$.

Proof The function f admits a (1, 1)-PBA, indeed, the mapping $A : (u,v) \mapsto f(u) + \nabla f(u)(v-u)$ satisfies assertions (a) and (b) in Definition 1.1 (see [9] for details).

It remains to show that lip $(f;\bar{x}) < \kappa^{-1}$. Since the Fréchet derivative of f is bounded on Ω, then f is Lipschitz continuous on Ω and it follows that lip $(f;\bar{x}) \le M < \kappa^{-1}$. We complete the proof by applying Theorem 3.5. □

4.2 A cubic method

When the function f is twice Fréchet differentiable near the point $\bar{x}$ and its second order derivative $\nabla^2 f$ is Lipschitz continuous on Ω, Geoffroy et al. [8] propose the following method to solve the inclusion $0 \in f(x) + F(x)$:

$$0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + \frac{1}{2}\nabla^2 f(x_k)(x_{k+1} - x_k)^2 + F(x_{k+1}). \tag{10}$$

They prove the existence of a sequence x_n which is cubically convergent to a solution of $0 \in f(x) + F(x)$ whenever $(f + F)^{-1}$ is Aubin continuous at $(0, \bar{x})$.

Corollary 4.2 *Assume that the sequence F_n is equi-metrically regular at $\bar{x}_n$ for $-f(\bar{x}_n)$ with a growth constant $\kappa > \sup_n \operatorname{reg} F_n(\bar{x}_n \mid -f(\bar{x}_n))$. If, in addition, the function f is twice Fréchet differentiable in an open neighborhood Ω of $\bar{x}$ and its second order derivative is Lipschitz continuous on Ω and if its derivative is bounded by a constant $M < \kappa^{-1}$ on Ω then, under assumptions* (A$_1$)–(A$_3$)*, there exist a sequence x_k generated by* (10) *and a positive constant C such that*

$$\|x_{k+1} - \bar{x}\| \le C\|x_k - \bar{x}\|^3,$$

i.e., the sequence x_k is cubically convergent to $\bar{x}$.

Proof The function f admits a (2, 1)-PBA, indeed, it is shown in [9] that the mapping $A : (u, v) \mapsto f(u) + \nabla f(u)(v - u) + \frac{1}{2}\nabla^2 f(u)(v - u)^2$ satisfies assertions (a) and (b). It is clear that the Lipschitz continuity of $\nabla^2 f$ on Ω ensures the boundness of ∇f on Ω. Thus the desired conclusion is a straightforward consequence of Theorem 3.5. □

Acknowledgement The authors thank the anonymous referees for providing constructive comments which improve the contents of this paper.

References

1. Aubin, J.-P., Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston (1990)
2. Beer, G.: Topologies on Closed and Closed Convex Sets. Kluwer Academic Publishers (1993)
3. Dontchev, A.L.: Local convergence of the Newton method for generalized equations, C. R. Acad. Sci., Paris, Ser. **1**, 327–329 (1996)
4. Dontchev, A.L., Hager, W.W.: An inverse mapping theorem for set-valued maps. Proc. Am. Math. Soc. **121**, 481–489 (1994)
5. Dontchev, A.L., Lewis, A.S., Rockafellar, R.T.: The radius of metric regularity. Trans. AMS **355**, 493–517 (2002)
6. Dontchev, A.L., Rockafellar, R.T.: Regularity and conditioning of solution mappings in variational analysis. Set-Valued Anal. **12**, 79–109 (2004)
7. Dontchev, A.L., Quincampoix, M., Zlateva, N.: Aubin criterion for metric regularity. (English summary) J. Convex Anal. **13**(2), 281–297 (2006)
8. Geoffroy, M.H., Hilout, S., Pietrus, A.: Acceleration of convergence in Dontchev's iterative method for solving variationnal inclusions. Serdica. Math. J. **22**, 385–398 (2003)
9. Geoffroy, M.H., Pietrus, A.: A general iterative procedure for solving nonsmooth generalized equations. Comput. Optim. Appl. **31**(1), 57–67 (2005)

10. Geoffroy, M.H., Hilout, S., Pietrus, A.: Stability of a cubically convergent method for generalized equations. Set-Valued Anal. **14**(1), 41–54 (2006)
11. Ioffe, A.D.: Metric regularity and subdifferential calculus. Uspekhi Matematicheskikh Nauk. **55**(3), 103–162 (2000); English translation in Russian Mathematical Surveys **55**, 501–558 (2000)
12. Mordukhovich, B.S.: Complete characterization of openness, metric regularity, and Lipschitzian properties of multifunctions. Trans. Am. Math. Soc. **340**(1), 1–35 (1993)
13. Pietrus, A. : Generalized equations under mild differentiability conditions. Rev. R. Acad. Cienc. Exact. Fis. Nat. **94**(1), 15–18, Matematicas (2000)
14. Robinson, S.M.: Regularity and stability for convex multivalued functions. Math. Oper. Res. **1**(2), 130–143 (1976)
15. Robinson, S.M.: Newton's method for a class of nonsmooth functions. Set-Valued Anal. **2**, 291–305 (1994)
16. Rockafellar, R.T., Wets, R.: Variational analysis. Ser. Comput. Stud. Math. **317**, Springer (1998)
17. Sonntag, Y., Zalinescu, C.: Set convergences. An attempt of classification. Trans. Am. Math. Soc. **340**(1), 199–226 (1993)
18. Ursescu, C.: Multifunctions with convex closed graph. Czechoslovak Math. J. **25**(100(3)), 438–441 (1975)